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MONOPHONIC DISTANCE

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Abstract

A $u\,v$ path is monophonic if it has no chords for any two vertices u and v in a connected graph G, and the monophonic distance $d_m(u,v)$ is the length of the longest $u\,v$ monophonic path in G. The monophonic eccentricity of each vertex v in G is given by $e_m(v) = max\ d_m(u,v): u\,V$. It is demonstrated that the monophonic center of a graph exists in every graph. The subgraph created by the vertices of G exhibiting minimal monophonic eccentricity is the monophonic center of G. Additionally, it is demonstrated that each connected graph G monophonic center is located within one of its blocks.

A R T I C L E I N F O

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1. INTRODUCTION

We take a finite connected graph with no loops and many edges, denoted by $G = (V(G), E(G))$. The letters p and q, respectively, stand for the order and size of G. The length of the shortest $u - v$ path in G is used to define the distance $d(u,v)$ between u and v.

The radius $rad\ G\ of\ G$ is the vertices of G is minimum. eccentricity, and the diameter $diam\ G\ of\ G$ is the maximum eccentricity among the vertices of G.

1.1. Definition. An edge $u_i u_j$ with $j \geq i + 2$ is a chord of a path $u_1, u_2, \dots, u_n$ in a connected graph G. If a $u - v$ path P is chord less, it is referred to as a monophonic path. The monophonic distance from $u\ to\ v$, abbreviated as $dm(u,v)$, is the length of the longest $u - v$ monophonic path. A $u - v$ monophonic path is defined as one whose length is equal to $dm(u,v)$.

1.2. Example. In $the\ graph\ G\ given\ in$ Figure 1.1. we can $easily$ check that $d(v_1, v_4) = 2, D(v_1, v_4) = 6$, and $d_m(v_1, v_4) = 4. The\ monophonic\ path\ P\ : v_1, v_2, v_8, v_7, v_4$ is $v_1 - v_4$ monophonic while the monophonic path $Q : v_1, v_3, v_4$ is not $v_1 - v_4$ monophonic.

The usual distance $d\ are\ metrics\ on\ the\ vertex\ set\ V\ of\ a\ connected\ graph\ G,\ whereas\ the\ monophonic\ distance\ dm$ not based on metrics V. To get the graph G given in Fig 1.1, $dm(v_4, v_6) = 5$, $dm(v_4, v_5) = 1$ and $dm(v_5, v_6) = 1$. Hence $dm(v_4, v_6) > dm(v_4, v_5) + dm(v_5, v_6), and\ so\ the\ triangle\ inequality\ is\ not\ satisfied$.

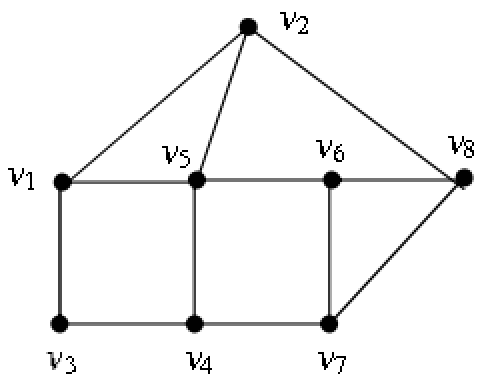

Figure 2.1.1

1.3. Result. *Let u and v be two vertices in a graph G. Then*

1. $dm(u,v) = 0$ if and only if $u = v$.
2. $dm(u,v) = 1$ if and only if uv is an edge of G
3. $dm(u,v) = p - 1$ *if and only if G is the path with endvertices u and v*
4. $dm(u,v) = dm(u,v) = D(u,v)$ *if and only if G is a tree.*

1.4. Definition. The monophonic eccentricity of each vertex v in a connected graph G is given by $e_m(v) = max\,\{d_m\,(u,v): u \in V$. A monophonic eccentric vertex of v is one where $d_m\,(u,v) = e_m\,(v)$ for the vertex u of G. The formulas $radm\,G = min\,\{e_m(v): v \in V\}$ and $diamm\,G = max\,\{e_m(v): v \in V\}$, respectively, determine the monophonic radius and diameter of G .

1.5. Example. We will use a condensed explanation in this example, as indicated in table 1.1. The graph G provided is shown in along with the vertices' eccentricities and monophonic distances in a monophonic manner. Note that $radm\,G = 3$ and $diamm\,G = 5$.

$dm\,(vi\,,vj\,)$	$v\,1$	$v\,2$	$v\,3$	$v\,4$	$v\,5$	$v\,6$	$v\,7$	$v\,8$	$em(v)$
$v1$	0	1	1	4	1	4	3	4	4
$v2$	1	0	4	3	1	5	4	1	5
$v3$	1	4	0	1	2	4	4	4	4
$v4$	4	3	1	0	1	5	1	4	5
$v5$	1	1	2	1	0	1	3	3	3
$v6$	4	5	4	5	1	0	1	1	5
$v7$	3	4	4	1	3	1	0	1	4
$v8$	4	1	4	4	3	1	1	0	4

Table 1.1. Figure 1.1 shows the monophonic eccentricities of the graph G vertices.

1.6. **Note.** In a tree T, there is only one path between any two vertices, u and v, and so $d(u,v) = dm(u,v) = D(u,v)$. *Hence* $rad\,T = radm\,T = radD\,T$ *and* $diam\,T = diamm\,T = diamD\,T$. Table 2.1.2 lists the monophonic diameter and monophonic radius of a few common graphs.

Graph G	K_p	C_p	$W_{1,p-1}(p \geq 4)$	$K_{1,p-1}(p \geq 2)$	$K_{m,n}(m,n \geq 2)$	P_n
$\mathrm{rad}_m\, G$	1	$p-2$	1	1	2	$\lfloor \frac{n}{2} \rfloor$
$\mathrm{diam}_m\, G$	1	$p-2$	$p-3$	2	2	$n-1$

Table 1.2. Several common graphs' monophonic diameter and radius

1.7. Theorem. (a) If a, b and c are $integers$ with $3 \leq a \leq b \leq c$, then a connected graph exists G such that $rad\ G = a, rad_m\ G = b\ and\ rad_D\ G = c$.

(b)If a, b and c are $integers$ with $5 \leq a \leq b \leq c$, then a connected graph appears. G exists in such a way that $diam\ G = a, diam_m\ G = b\ and\ diam_D\ G = c$.

Proof. (a) Three examples are used to demonstrate the conclusion**.**

Case (i) $3 \leq a = b = c.\ Consider.\ G = P_{2a+1}$, path,of order $2a + 1$. It is clear that $rad\ G = rad_m\ G = rad_D\ G = a$.

"**Case (ii)** $3 \leq a \leq b < c$. Let $F_1 : u_1, u_2, \dots, u_{a-1}$ and $F_2 : v_1, v_2, \dots, v_{a-1}$ be two copies of the path P_{a-1} of order a – 1. Let $F_3 : w_1, w_2, \dots, w_{b-a+3}$ and $F_4 : z_1, z_2, \dots, z_{b-a+3}$ two duplicates of the path P_{b-a+3} of order $b - a + 3$, and $F_5 = K_{c-b+1}$ the complete graph of order $c - b + 1$ with $V(F_5) = \{x_1, x_2, \dots, x_{c-b+1}\}$. We construct the graph G as follows : (i) identify the vertices x_1 in F_5 and w_1 in F_3 ; also identify the vertices x_{c-b+1} in F_5 and z_1 in F_4 ; (ii) identify the vertices w_{b-a+3} in $F_3 and\ u_2\ in\ F_1$, and identify the vertices z_{b-a+3} in F_4 and v_2 in F_2 ; and (iii) join each vertex w_i $(1 \leq i \leq b - a + 2)$ in F_3 and u_1 in F_1, and join each vertex z_i $(1 \leq i \leq b - a + 2)$ in F_4 and v_1 in F_2. The resulting graph G is shown in Figure 1.2. It is easily verified that $e(v) = a$ if $v \in V(F_5)$; $e(v) > a$ if $v \in V(G) - V(F_5), e_m$(v) = $b\ if\ v \in V(F_5)$; $e_m(v) > b\ if\ v \in V(G) - V(F\_5)\ and\ e\_D\ (v) = c\ if\ v \in V(F5)$; and $e_D(v) > c\ if\ v \in V(G) - V(F_5)$. It follows that $rad\ G = a, rad_m\ G = b, and\ rad_D\ G = c$"

"**Case (iii)** $3 \leq a < b = c. Let\ E_1 : v_1, v_2, \dots, v_{2a+1}$ be a path of order $2a + 1. Let\ E_2 : u_1, u_2, \dots, u_{b-a+3}$ and $E_3 : w_1, w_2, \dots, w_{b-a+3}$ be two copies of the path P_{b-a+3} of order $b - a + 3$, and $let\ E_i\ (4 \leq i \leq 2\,(b - a) + 3)$ be $2(b - a)$ copies of K_1. We construct the graph G as follows : (i) identify the vertices v_{a+1} in $E_1, u_1\ in\ E_2$, and $w_1\ in\ E_3$; (ii) identify the vertices v_{a-1} in E_1 and $u_{b-a+3}\ in\ E_2$, and identify the vertices $v_{a+3}\ in\ E_1$ and w_{b-a+3} in E_3 ; and (iii) join each E_i $(4 \leq i \leq b - a + 3)$ with v_{a+1} in E_1 and $ui - 1_{i-1}\ in\ E_2$, and join each E_i $(b - a + 4 \leq i \leq 2(b - a) + 3)$ with v_{a+1} in E_1 and $w_{i-b+a-1}$ in E_3. The final graph, G, is displayed in Figure 1.3."

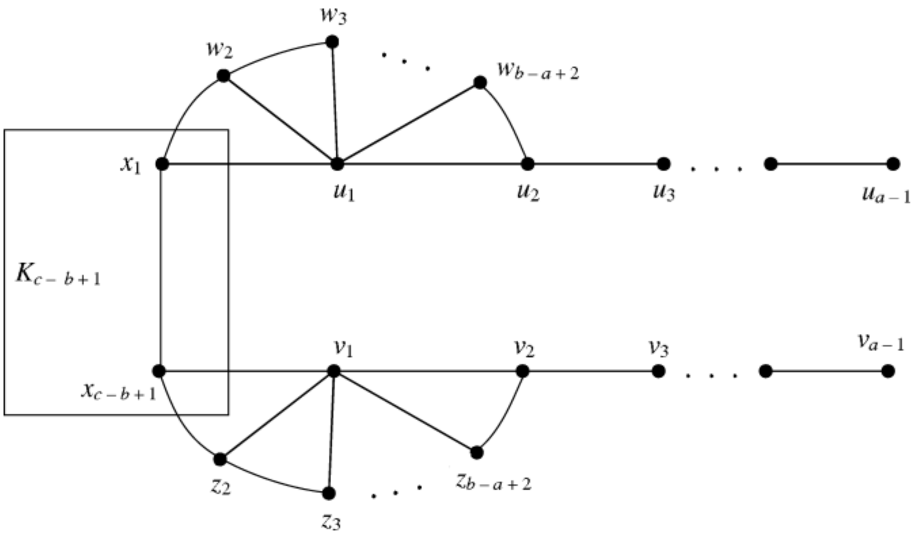

Fig 1.2. a.graph G in $Case$(ii) of.Theorem(a).

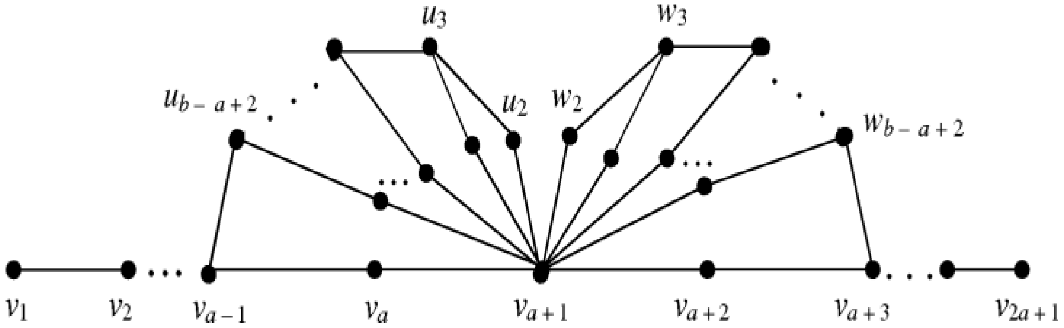

Fig 1.3. a.graph G in $Case$(iii) of.Theorem(a).

It is simple to confirm that $e(v_{a+1}) = a$; $e(v) > a \; if \; v \in V(G) - \{v_{a+1}\}$; $e_m(v_{a+1}) = b$; $e_m(v) > b \; if \; v \in V(G) - \{v_{a+1}\}, and \; e_D(v_{a+1}) = c$; $and \; e_D(v) > c \; if \; v \in V(G) - \{v_{a+1}\}$. It follows that $rad\, G = a, rad_m\, G = b, and \; rad_D\, G = c$.

(b) By taking into account three situations, this result is also demonstrated.

Case (i) $5 \leq a = b = c$. Let G be a path of.order $a + 1$. Then $diam\, G = diam_m\, G = diam_D\, G = a$.

Case (ii) $5 \leq a \leq b < c$ let $F_1: u_1, u_2, \ldots, u_{a-1}$ be the.path P_{a-1} of order $a - 1$; $F_2: w_1, w_2, \ldots, w_{b-a+3}$ be the.path P_{b-a+3} of.order $b - a + 3$; and $F_3 : K_{c-b+1}$ be the full graph.of order The graph G is created as follows : I Identify the.vertices w_{b-a+3} in F_2 and u_2 in F_1 ; (ii) join each.vertex w_i $(1 \leq i \leq b - a + 2)$ in F_2 and u_1 in F_1 ; and (iii) identify the vertices x_1 in F_3 and w_1 in F_2; Figure 2.1.4 displays the final graph G.

It is easily verified that $e(v) = a \; if \; v \in (V(F_3) - \{x_1\}) \cup \{u_{a-1}\}$; $e(v) < a \; if \; v \in V(F_2) \cup (V(F_1) - \{u_{a-1}\})$, and $e_m(v) = b \; if \; v \in (V(F_3) - \{x_1\}) \cup \{u_{a-1}\}$; $e_m(v) < b \; if \; v \in V(F_2) \cup (V(F_1) - \{u_{a-1}\}), and \; e_D(v) = c \; if \; v \in (V(F_3) - \{x_1\}) \cup \{u_{a-1}\}$; $and \; e_D(v) < c \; if \; v \in V(F_2) \cup (V(F_1) - \{u_{a-1}\}). It \; follows \; that \; diam\, G = a, diam_m\, G = b \; and \; diam_D\, G = c.$

Case (iii)" $5 \leq a < b = c. Let \; E_1 : v_1, v_2, \ldots, v_{a+1}$ be a path of order $a + 1$; $E_2 : w_1, w_2, \ldots, w_{b-a+3}$ be another path of order $b - a + 3$; $and \; E_i \; (3 \leq i \leq b - a + 2) \; be \; b - a$ copies of K_1. Let G be the graph obtained from $E_i \; for \; i = 1, 2, \ldots, b - a + 2$ by identifying the vertices v_{a-2} and v_a of E_1 with w_1 and w_{b-a+3} of E_2, respectively, and joining each E_i $(3 \leq i \leq b - a + 2)$ with v_{a-2} and w_i. The graph G is shown in Fig 1.5".

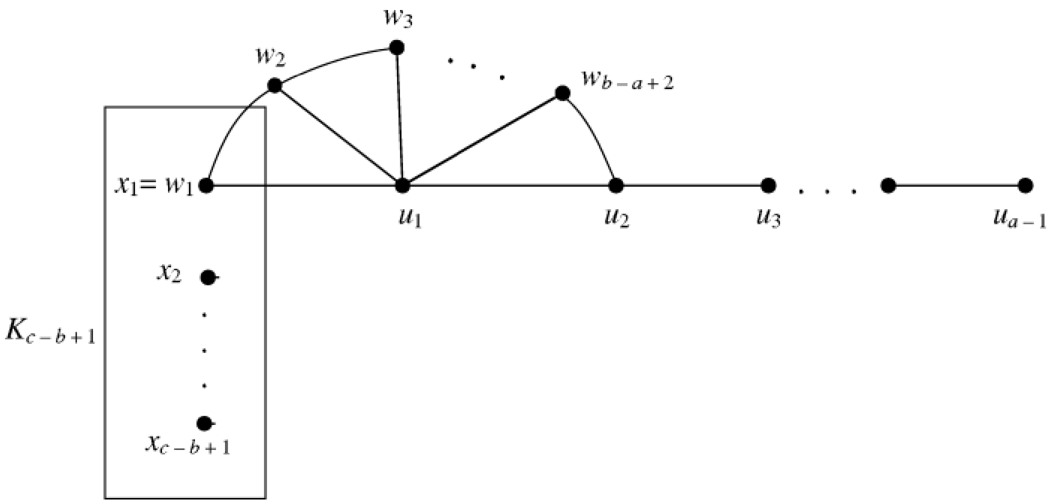

Fig 1.4. a.graph G in Case(ii) of.Theorem(b).

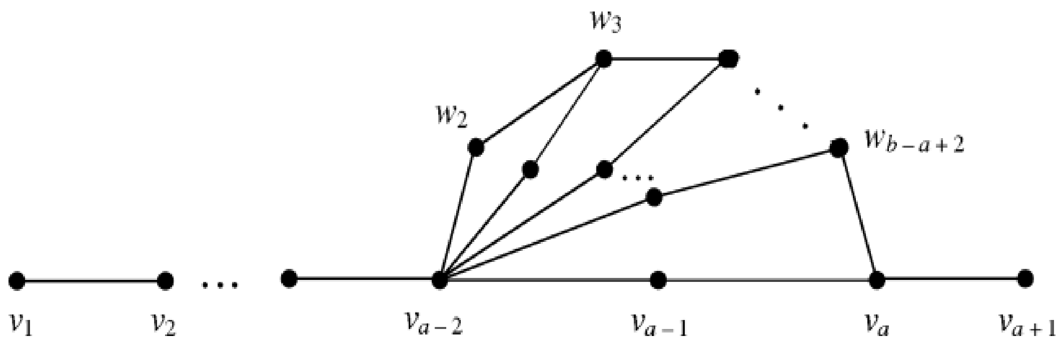

Fig 1.5. a graph G in Case(iii) of Theorem(b)

It is easily verified that $e(v) = a$ if $v \in \{v_1, v_{a+1}\}$; $e(v) \leq a\ if\ v \in V(G) - \{v_1, v_{a+1}\}, and\ e_m(v) = b\ if\ v \in \{v_1, v_{a+1}\}$; $e_m(v) \leq b\ if\ v \in V(G) - \{v_1, v_{a+1}\}, and\ e_D(v) = c\ if\ v \in \{v_1, v_{a+1}\}$; and e_D(v) ≤ c if $v \in V(G) - \{v_1, v_{a+1}\}$. It follows that $rad\ G = a, rad_m\ G = b$ and $rad_D\ G = c$. The inequality $rad\ G \leq diam\ G \leq 2\ rad\ G\ and\ rad_D\ G \leq diam_D\ G \leq 2\ rad_D\ G$ hold for any connected graph G. This is not applicable to monophonic radius and monophonic diameter, though. For instance, it is clear that $rad_m G = 1$ and $diam_m G = p - 3 \geq 3$ so that $diam_m\ G > 2\ rad_m\ G$ when the.graph G is the wheel $W_{1,p-1}$ $(p \geq 6)$. that there is a connected graph G with $rad\ G = a$ and $diam\ G = b$ If a and b are any two consecutive positive $integers$, then $a \leq b \leq 2a$. that there is a connected.graph G with $rad_D\ G = a$ and $diam_D\ G = b$ If a and b are any two consecutive positive $integers$, then $a \leq b \leq 2a$.

The theorem that follows now provides a realization result for $rad_m\ G\ diam_m\ G$.

1.8. Theorem. There exists a connected graph G such that $rad_m\ G = a$ and $diam_m\ G = b$ if $a\ and\ b$ are positive integers with $a \leq b$.

Proof. Three cases are used to demonstrate this result.

Case(i) $a = b \geq 1$. Let G be the.cycle C_{a+2}. Then $rad_m G = a$ and $diam_m G = b$

Case (ii) $a < b \leq 2a. Let\ C_1 : u_1, u_2, \ldots, u_{a+2}, u_1$ be a cycle of order, and $C_2 : v_1, v_2, \ldots, v_{b-a+2}, v_1$ be a cycle of order, respectively. The graph that results from finding the vertex(es) $u_1\ of\ C_1\ and\ v_1\ of\ C_2$ is denoted by G. Since $b \leq 2a, b - a + 2 \leq a + 2$ follows naturally. It is obvious that for any $x\ in\ G, d_m(u_1, x) = a\ and\ d_m\ (u_1, u_{a+1}) = a$, and as a result, $e_m\ (u_1) = a$. Furthermore, it is clear that $rad_m\ G = a$ because no vertex in G has $e_m\ (x) < a$, and $e_m(u_3) = b$ because it is clear that $d_m(u_3, v_3) = b\ and\ d_m(u_3, x) \leq b$. Additionally, it is clear that for each vertex $x\ in\ G, e_m(x) \leq b$, resulting in $diam_m\ G = b$.

Case (iii) $b > 2a$. Let G stand for the graph formed by finding the end vertex of the path, P_{2a}, and the wheel's center vertex, $W = K_1 + C_{b+2} (b \geq 2)$. Since $b > 2a$, each vertex of $x \in V(C_{b+2})$ because $e_m(x) = b$. Additionally, every vertex of $x \in V(P_{2a})\ and\ e_m(v_a) = a$. As a result, $rad_m\ G = a\ and\ diam_m\ G = b$.

1.9. Remark. For $integers\ a$ and b with $2a < b$, each vertex in the graph G given in Fig 1.6. has monophonic eccentricity b or $n(a \leq n \leq 2a)$. So, unlike standard eccentricity, if k is an integer such that $rad_m G < k < diam_m G$, there may not be a vertex x of G such that $e_m(x) = k$.

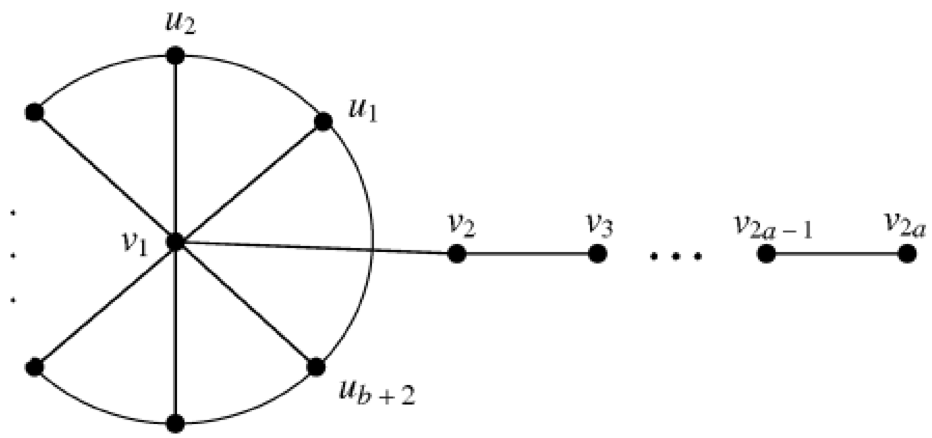

Figure 1.6

2. Monophonic.center and. monophonic periphery

2.1. Definition. The monophonic center $C_m(G)$ $of\ G$ is the subgraph that is generated by the G single-note center vertices. If e_m (v) = rad_m G, a vertex v in a connected graph G is referred to as a monophonic central vertex. The monophonic periphery is the subgraph that G monophonic peripheral vertices form.

$P_m\ (G) of\ G. If\ e_m(v)\ =\ diam_m\ G$, a vertex $v\ in\ G$ is referred to as a monophonic peripheral vertex.

2.2.R emark. It is not necessary for a connected graph's monophonic center to be connected. $C_m(G) = \{v_3, v_6\}$. in relation to the graph G in Figure 2.1.

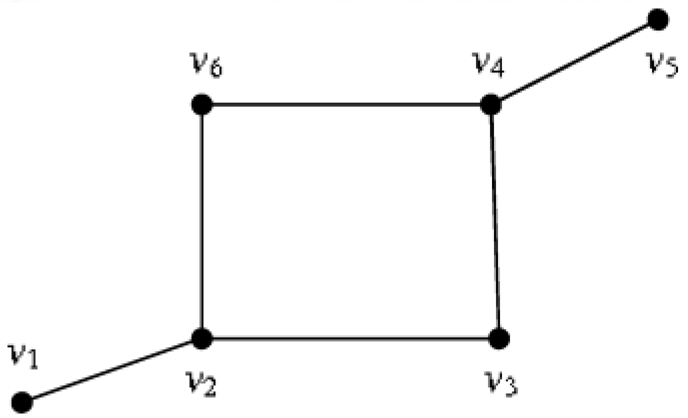

Figure2.1. a.graph G in.Remark

2.3. Theorem. Every.graph has a connected.monophonic.center.

Proof. G should be a graph. We demonstrate how G is a graph's monophonic center. Let the monophonic diameter of G be given $by\ l\ =\ d_m$. Let $P\ :\ u_1, u_2, \ldots, u_i$ and $Q\ : v_1, v_2, \ldots\ldots, v_i$ be two copies of the path Pl. By connecting each vertex of graph G with $u_1\ in\ P\ and\ v_1\ in\ Q$, the necessary graph H shown in Figure 2.2 is obtained from graphs $G, P, and\ Q$. Then, for each vertex $x\ in\ G, e_{mH}(x) =\ d_m$, and for each vertex x outside of $G, d_m + 1 \leq\ e_{mH}\ (x)\ \leq\ 2\ d_m . C_m\ (H)\ =\ G$ because $V(G)$ is the collection of monophonic central vertices of H.

2.4. Theorem. Every connected graph's monophonic center $C_m\ (G)$. Some block of G is a subgraphs of G.

Proof. Assume a connected graph G exists with a monophonic center $C_m\ (G)$ that is not a subparagraph of a G block.. Afterward, G has a cut vertex v. resulting in $G\ -\ v$ having two components, H_1 and H_2, each with $C_m(G)$ vertices. Let u be a G vertex such that $e_m\ (v)\ =\ d_m\ (u, v)$, and P_1 be the longest $u - v$ monophonic path in G. Consequently, at least one of H_1 and H_2 lacks P_1 vertices, for example, H_2. Now consider a vertex

w in $C_m\ (G)$ that belongs to H_2, and consider P_2 to be the longest $v - w$ monophonic path in G .P_1 followed by P_2 yields the $u - w$ longest.monophonic path with a length.greater than P_1 because v is a cut vertex. This results in $e_m\ (w)\ >\ e_m\ (v)$, implying the contradiction that w is not the monophonic central vertex of G.

2.5.Problem. Considering any three positive integers a, b, and $c\ with\ 1\ \leq\ a\ \leq\ b\ \leq\ c$ whether a connected graph G exists $diam\ G\ =\ a, diam_m G\ =\ b\ and\ diam_D G\ =\ c$?

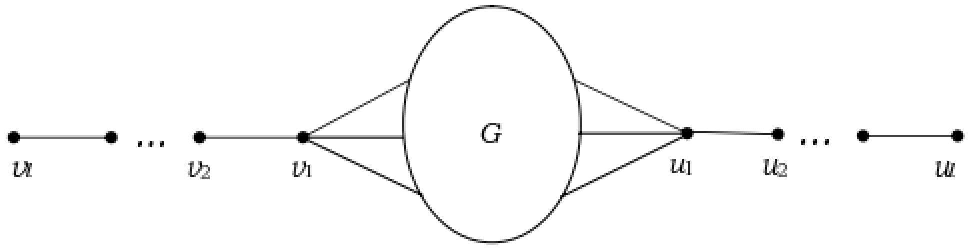

Figure 2.2

Solution: We consider the following four instances.

Case 1. $a\ =\ 1$ If such a graph exists, G is a complete graph of order $c\ +\ 1$ for some $c\ \geq 1$ because diam $G\ =\ 1$. Therefore, $1\ =\ a\ =\ b \leq\ c$ and $b\ =\ diam_m\ G\ =\ 1$ and $diam_D\ G\ =\ c$. For some $c \geq 1$, however, G is a complete graph of order $c\ +\ 1$ if $a\ =\ b\ =\ 1$. As a result, if and only if $1\ =\ a\ =\ b\ \leq c$, there is a graph G with diam $G\ =\ a\ =\ 1, diam_m\ G\ =\ b, and\ diam_D\ G\ =\ c$.

Case 2. $a\ =\ b\ =\ c$.

A desired graph is one with a path of order $c + 1$. (In reality, $diam\ T\ =\ diam_m T\ =\ diam_D T$ is a tree T property.)

Case 3. $2\ \leq\ a\ \leq\ b\ <\ c$.

Let a path lead to the graph $G, P\ :\ u_0, u_1, u_2, \ldots, u_C$ by joining the vertices u_c and $u_t\ for\ a\ -\ 2\ \leq\ t\ <\ c$, and u_i and u_j for $b\ -\ 1\ \leq\ i < j\ \leq\ c$ (avoiding the multiple edges formed during the construction). It is routine to check that $diam\ G\ =\ a, diam_m G\ =\ b, and\ diam_D G\ =\ c$.

Case 4. $2\ \leq\ a < b\ =\ c$

First, suppose $2\ \leq\ a\ \leq\ 3$. Let $P\ :\ u_0, u_1, u_2, \ldots, u_C$ be a monophonic path of length c. Since $a\ <\ c, P$ is not a $u_0 -\ u_c$ geodesic. Let $Q\ :\ u_0, v_1, v_2, \ldots, v_k$, uc be a $u_0 -\ u_c$ geodesic. Since P is monophonic, $v_1\ =\ u_i$ for $2\ \leq\ i\ \leq\ c$. Moreover $v_1\ =\ u_1$. Otherwise, $P_1\ :\ v_1, u_0, u_1, u_2, \ldots, u_c$ is a path of length $c\ +\ 1$, which is a contradiction. Similarly, we have $v_k\ =\ u_{c-1}$. By the same argument as above, we may assume that $v_i\ =\ u_i$ for $1\ \leq\ i\ \leq\ s\ or\ t\ \leq\ i\ \leq\ c\ -\ 1$, where $s < t$ and $v_j\ =\ u_j$ for $j\ =\ s\ +\ 1\ or\ t\ -\ 1$. Hence, $d(u_0, u_c)\ \geq\ 4\ \geq\ a\ +\ 1$, which is a contradiction. Therefore, no such graphs exist in this subcase

Let's say that's $a \geq 4$ now. We can create the graph G from the path P: $u_0, u_1, u_2, \ldots u_c$ by adding a new vertex v and connecting it to the vertices $u_{c-a+3}\ and\ u_{2i-1}\ for\ 1\ \leq 2i\ -\ 1 <\ c\ -\ a\ +\ 2$. Verifying that $diam\ G\ =\ a, diam_m\ G\ =\ b$, and $diam_D\ G\ =\ c$ is routine.

2.6. Theorem. "A non-trivial graph G is the monophonic periphery of some con nected graph if and only if every vertex of G has monophonic eccentricity 1 or no vertex of G has monophonic eccentricity 1".

Proof. "Suppose that every vertex of G has monophonic eccentricity 1. Then $P_m(G)\ =\ G$. Next, suppose that no vertex of G has monophonic eccentricity 1. Hence for any vertex x in G, there is a vertex y in G such that $e_m(x)\ =\ d_m(x, y)\ \geq\ 2$.

Clearly, $e_m(x)\ \leq\ p\ -\ 1$. Now, take p vertex disjoint paths $P_i(1\ \leq\ i\ \leq\ p)$ each of length $p\ -\ 1\ such$ that no vertex of P_i is a vertex of G. Identify the end vertices of one path, say P_i, with $x\ and\ y$, thereby producing a cycle of length $e_m(x)\ +\ p - 1$ This is done for every vertex $z\ =\ x\ of\ G$ by taking a

path $P_j(i = j)$. Let the graph obtained be G_1. Now, for every path $P_i(1 \leq i \leq p)$ in G_1, join each internal vertex of P_i with every vertex of $V(G_1) - V(P_i)$, avoiding multiple edges. Let H be the resulting graph obtained. (It is to be noted that if y is a monophonic eccentric vertex of x, then x is also a monophonic eccentric vertex of y, and adjoining a path as mentioned above, may or may not be done. This does not affect the monophonic eccentricity of any vertex in H.) Let $e_mH(v)$ denote the monophonic eccentricity of a vertex $v\ in\ H$. Then it is clear that $e_mH(v) = p - 1$ for any vertex $v\ in\ G$ and $e_mH(v) \leq p - 2$ for any vertex v not in G. Hence $P_m(H) = G$. The graph in Fig. 9 shows the construction of the graph H when G is the path v_1, v_2, v_3, v_4, where $e_mH(v) = 3$ for every vertex $v\ in\ G$ and $e_mH(v) = 2$ for every vertex $v\ not\ in\ G$.

Conversely, let $G = P_m(H)$. Suppose that some but not all vertices of G have monophonic eccentricity 1. Certainly G is a proper subgraph of H. Therefore, for each vertex $x\ of\ G$, it follows that $e_mH(x) = diam_m\ H \geq 2$. Let u be a vertex of G having monophonic eccentricity $1\ in\ G$. Then, u is adjacent to all other vertices of G. Let v be a vertex of H such that $d_mH(u,v) = e_mH(u) = diam_m\ H \geq 2$. Hence $e_mH(v) = diam_m\ H$ and so $v \in P_m(H) = G$. Hence $u\ and\ v$ are adjacent in G and so $u\ and\ v$ are also adjacent in H so that $d_mH(u,v) = 1$, which is a contradiction".

2.7. Definition. If $rad_m\ G = diam_m\ G$, or if G is its own monophonic center, a connected graph G is monophonic and self-centered .

3. Monophonic number of a graph

3.1. Definition. If each vertex v of a graph G lies on an $x - y$ monophonic path in G for some $x, y \in S$, then the set S of its vertices is said to be a monophonic set of G. The monophonic number is the minimum cardinality of a monophonic set of G.

and is indicated by of $G.\ m(G)$.

3.2. Example. The minimum monophonic sets of the graph G shown in Figure 3.1 are $S_1 = \{x, w\}$ and $S_2 = \{u, w\}$, and as a result, $m(G) = 2$.

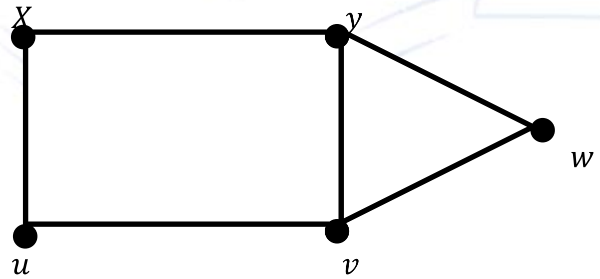

Figure 2.3.1 . A graph G with $m(G) = 2$

If a vertex v in a graph G is a member of each minimal monophonic set in G, then it is a monophonic vertex. Every vertex in S is a monophonic vertex if G has a singular minimal monophonic set S. In the following theorem, we demonstrate that a nontrivial linked graph G has certain vertices that are monophonic G vertices.

3.3. Theorem Every extreme vertex of a connected graph G is contained in every monophonic set of the graph... Additionally, S is the specific minimum monophonic set of G if the set S of all extreme vertices of G is a monophonic set.

Proof. Let S be a monophonic set of G and let u be an extreme vertex. Assume that $u \notin S$ Then, for some $x, y \in S, u$ is an internal vertex of a $x - y$ monophonic path, let say P. Allow v and w to be u 's neighbors on P. This results in a contradiction because v and w are not contiguous and u is not an extreme vertex. Consequently, u is a member of every monophonic set of G.

3.4. Corollary. For complete graph $K_p(p \geq 2), m(K_p) = p$.

3.5. Theorem .Let S be a monophonic set of G and let G be a connected graph with a cutvertex named v. Then, an element of S is contained in each component of $G - v$.

Proof. Consider a component B of $G - v$ which does not contain any vertex of S. Let any vertex in B be you. Due to the fact that S is a monophonic set, there is a pair of vertices x and y in S such that u lies in some $x - y$ monophonic path $P: x = u_0, u_1, u_2, \ldots, u, \ldots, u_n = y$ in G with $u \neq x, y. V$ being a cutvertex of Both the $u - y$ subpath P_2 of P and the $x - u$ subpath P_1 of P contain v, it. Hence, which is a contradiction, P is not a path.

3.6. Theorem .A connected graph G cutvertex does not belong to any minimum monophonic set of G.

Proof. Let S be the minimum monophonic set of G and let v be a cutvertex of G. Theorem 2.5 states that every part of $G - v$ contains a part of S. Let U and W be two separate parts of $G - v$, where $u \in U$ and $w \in W$. Following that, v is an internal vertex of a monophonic path $u – w$. Let $S' = S - \{v\}$. Every vertex that is located on an $u - v$ monophonic path is evidently also Every vertex that is located on an $u - v$ monophonic path is evidently also monophonic set of G, which is in contrast to the statement that S is a minimum monophonic set of G.

3.7. Theorem .If G is a connected non-complete graph with a minimum cutset of vertices, then $m(G) \leq p - k$

Proof. G is an non-complete connected graph, hence it is obvious that $1 \leq k \leq p - 2$. Let U be the minimum cutset of G, where $U = \{u_1, u_2, u_3, \ldots, u_k\}$. Let $S = V - U$ and $G_1, G_2, \ldots, G_r, (r \geq 2)$ be the parts of $G - U$. Then, for each j $(1 \leq j \leq r)$, every vertex $u_i (1 \leq i \leq k)$ is close to at least one vertex of G_j. Since S is obviously a monophonic set of G, $m(G) \leq |S| = p - k$.

3.8. Remark. Theorem 2.3.7 has a sharp bound. For the cycle $C_4, m(C_4) = 2$. Also $\kappa = 2$ and $p - \kappa = 2$. Thus $m(G) = p - \kappa$

3.9. Theorem : G is complete If and only if $m(G) = p$, for any connected.graph G of order p

Proof. Suppose $m(G) = p$. Assume that G is not.a fully complete graph. Then.there are two.vertices $u \text{ and } v$ that are such that they are not next to one another in G. G is connected, hence there is a monophonic path with length at least 2 from u to v, let x say P. In order for $x \neq u, v$, it must be a vertex of P. Therefore, $m(G) \leq p - 1$ is incongruent since $S = V - \{x\}$ is a monophonic set of G.

3.10. Definition : Choose any vertex in G to represent x. If any vertex z with $d_m(x, y) < d_m(x, z), z$ lies on an $x - y$ monophonic path, then vertex y in G is said to be an $x -$ monophonic superior vertex.

3.11. Theorem. Let x represent any G vertex. Then, each x - monophonic superior vertex is a monophonic eccentric vertex of x.

Proof. So that $e_m(x) = d_m(x, y)$, let y be a monophonic eccentric vertex of x. There exists a vertex z in G such that $d_m(x, y) < d_m(x, z)$ and z does not reside on any $x - y$ monophonic path, leading to the contradiction that $e_m(x) \geq d_m(x, z) > d_m(x, y)$, which occurs if y is not an x - monophonic superior vertex.

3.12. Note .Theorem 3.11 converse is untrue. The cycle C_6 has the following vertices: $v_1, v_2, v_3, v_4, v_5, v_6, v_1$, where v_4 is a v_1 - monophonic superior vertices and not a v_1 - monophonic eccentric vertices.

3.13. Theorem .Supposing G is a connected graph, If and only if two vertices x and y exist, with y being an x-monophonic superior vertex and every vertex of G being on an $x - y$ monophonic path, then $m(G) = 2$.

Proof.Assume that $S = \{x, y\}$ is a minimum monophonic set of G and that $m(G) = 2$. There is a vertex z in G with $d_m(x, y) < d_m(x, z)$ and z does not reside on any $x - y$ monophonic path if y is not an x-monophonic superior vertex. This results in a contradiction because S is not a monophonic set of G.

4. Bounds for the monophonic number of a graph

We provide an improved upper bound for the monophonic number of a graph in the following theorem in terms of its order and monophonic diameter. We use the term "d_m" to represent the monoponic diameter diammG for convenience.

4.1. Theorem .$m(G) \leq p - d_m + 1$ if G is a non-trivial connected graph with order p and monophonic diameter d_m.

Proof .Let $P: u = v_0, v_1, \ldots, vd_m = v$ be an $u - v$ monophonic path of length d_m. Let u and v be the vertices of G such that $d_m(u, v) = d_m$. Let $S = V - \{v_1, v_2, \ldots, vd_m - 1\}$. When $m(G) \leq |S| = p - d_m + 1$, it is evident that S is a monophonic set of G. In order to ensure that the bound in Theorem 2.4.1 is sharp, for the complete graph $K_p(p \geq 2), d_m = 1$ and $m(K_p) = p$.

4.2. Theorem. $2 \leq m(G) \leq g(G) \leq p$ for each connected graph G of order p.

Proof. Every geodesic is a monophonic path, hence every geodetic set must also be a monophonic set. Consequently, $m(G) \leq g(G)$. The other disparities are trivial.

4.3. Remark 3.1.Theorem 4.2 bounds are exact. Assuming that K_p is a complete graph, $m(K_p) = g(K_p) = p$. $m(P_n) = g(P_n) = 2$ for the path P_n, which is non-trivial. Additionally, $m(G) = g(G)$ is a complete bipartite graph, an even cycle, or a non-trivial tree (G). In Theorem 4.2, every inequality is a rigorous inequality. $S = \{v_6, v_7, v_3\}$ is a minimum monophonic set of the graph G shown in Figure 4.1 such that $m(G) = 3$ and no 3-elements subset of the vertex set is a geodetic set of G. A geodetic set of G is $S \cup \{v1\}$, hence it follows that $g(G) = 4$. As a result, we have $2 < m(G) < g(G) < p$.

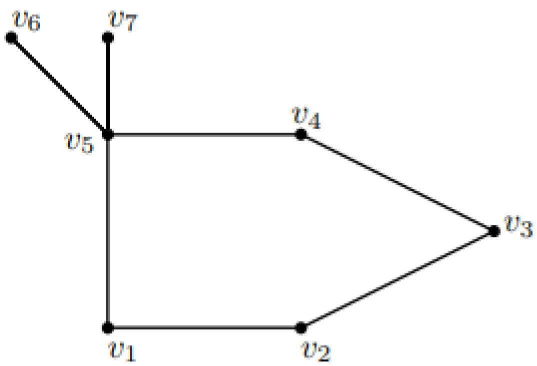

Figure 4.1. A graph G in Remark 4.3. with $2 < m(G) < g(G) < p$

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